MATH 147 QUIZ 10 SOLUTIONS

1. Use the surface area formula to find the surface area of an open cylinder whose base has radius r and whose height is h. What is the surface area if the cylinder is closed, i.e, has a top and bottom? (5 Points)

   The surface area formula for a surface with two parameters u, v is $\iint_S \|T_u \times T_v\| \, dS$. This cylinder can be written as $g(u, v) = (r\cos(v)\vec{i} + r\sin(v)\vec{j} + u\vec{k})$. We note that the bounds are $0 \leq u \leq h$, and $0 \leq v \leq 2\pi$. We have $T_u = \vec{k}$, and $T_v = -r\sin(v)\vec{i} + r\cos(c)\vec{j}$. Then, we have the cross product as

$$T_u \times T_j = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -r\sin(v) & r\cos(v) & 0 \end{vmatrix} = -r\cos(v)\vec{i} - r\sin(v)\vec{j}.$$

   We take the magnitude of the cross product, giving $\|T_u \times T_v\| = \sqrt{r^2\cos^2(v) + r^2\sin^2(v)} = r$. Finally, we compute the double integral.

$$\int_0^{2\pi} \int_0^h r \, du \, dv = 2\pi rh.$$

   As for the closed cylinder, a computation done in class or geometry knowledge tells us that the area of the top and the bottom is πr^2 each, and so the total surface area of the closed cylinder is $2\pi rh + 2\pi r^2$.

2. Evaluate $\iint_S x^2 + y^2 + z^2 \, dS$, where S is the portion of the plane $z = x + 1$ that lies in the cylinder $x^2 + y^2 = 1$. (5 points)

   We begin by parametrizing S. First, z is given in terms of x, and our first description of the surface is $(x, y, x + 1)$. However, this parametrization is difficult to work with due to our bounds being the cylinder. We thus move to polar coordinates, and we write S as $u\cos(v)\vec{i} + u\sin(v)\vec{j} + (u\cos(v) + 1)\vec{k}$. Now, we have $0 \leq u \leq 1$, and $0 \leq v \leq 2\pi$. We find the partials, with $T_u = \cos(v)\vec{i} + \sin(v)\vec{j} + \cos(v)\vec{k}$, and $T_v = -u\sin(v)\vec{i} + u\cos(v)\vec{j} - u\sin(v)\vec{k}$. This gives

$$\begin{aligned} T_u \times T_v &= \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & \cos(v) \\ -u\sin(v) & u\cos(v) & -u\sin(v) \end{vmatrix} \\ &= (-u\sin^2(v) - u\cos^2(v))\vec{i} - (-u\cos(v)\sin(v) + u\cos(v)\sin(v))\vec{j} + (u\cos^2(v) + u\sin^2(v))\vec{k} \\ &= -u\vec{i} + u\vec{k}. \end{aligned}$$

Thus, we have $\|T_u \times T_v\| = \sqrt{2}u$. We now integrate, and we get

$$\begin{aligned}
\iint_S x^2 + y^2 + z^z \ dS &= \int_0^1 \int_0^{2\pi} (u^2 + (u\cos(v) + 1)^2)(\sqrt{2}u) \ dv \ du \\
&= \sqrt{2} \int_0^1 \int_0^{2\pi} u^3 + u^3\cos^2(v) + 2u\cos(v) + u \ dv \ du \\
&= \sqrt{2} \int_0^1 2\pi u^3 + 2\pi u + u^3\pi \ du \\
&= \sqrt{2} \int_0^1 3\pi u^3 + 2\pi u \ du \\
&= \sqrt{2}(\frac{3\pi}{4} + \pi) = \frac{7\pi\sqrt{2}}{4}.
\end{aligned}$$